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Entangled States, Holography, and Quantum Surfaces

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Entangled States, Holography, and Quantum Surfaces

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Starting with an elementary discussion of quantum holography, we show that entangled quantum states of qubits provide a "local" representation of the global geometry and topology of quantum Riemann surfaces. This representation may play an important role in both mathematics and physics. Indeed, the simplest way to represent the fundamental objects in a "theory of everything" may be as muti-qubit entangled states.

1. Introduction

It has recently been suggested [1] that the whole of mathematics may be unified with theoretical physics by virtue of the fact that there is a close connection between multi-channel quantum mechanics and the geometry of quantum surfaces. This connection first appeared in the theory of adaptive optics in the presence of photon noise [2], and subsequently has reappeared in various disguises: "quantum cohomology" [3], a quantum theory of membranes [4], and quantum holography [5]. This paper is intended as an elementary introduction to the relationship between these subjects and ordinary quantum mechanics. In particular, we would like to draw attention to the connection between quantum manipulations of entangled states of qubits and the introduction of deformable surfaces as quantum objects.

In section 2 we approach the problem of quantizing a curved two dimensional surface via its holographic representation. We begin section 2 by introducing a paraxial-ray type approximation for the holographic representation. The usual mode variables for the electromagnetic field are replaced by field variables $\mathbf{E}$ and $\mathbf{E}^+$ whose transverse variation represents the structure of a hologram. These field variables depend on both the position on the hologram and the orientation of the illuminating laser beam, and quantum fluctuations of these variables reflect quantum "fuzziness" in the surface. This may perhaps be interpreted by saying that the classical surface has been replaced with a "quantum surface" [6]. In section 3 we show that this quantum surface can also be described as a family of classical surfaces, and therefore corresponds to a certain kind of moduli space. A natural question that arises in this connection is how to characterize the moduli space associated with a non-classical hologram. In section 4 this question is considered for the case where the surface in question is a complex curve; i.e. a Riemann surface

Just as is the case with classical holographic reconstructions of a reentrant surface, a central problem for quantum holography is how to represent the global topology of the surface. However, in contrast with the classical situation, it appears that in the case of quantum holography there is, at least conceptually, a very elegant way of representing the global topology of the surface. Namely, the global topological structure of a quantum surface can be represented by forming a quantum superposition of quantum holographic representations corresponding to different views of the surface. In section 5 we comment on this remarkable possibility for the case of a Riemann surface. As it turns out entangled quantum holographic representations of a Riemann surface provide a pedestrian introduction to some very sophisticated mathematical structures. In addition such representations may be the long sought fundamental degrees of freedom in a theory of everything.

2. Non-classical holographic representations

As a quick reminder all information concerning an arbitrarily curved surface in 3-dimensions can be encoded onto a flat plane or the surface of a sphere by recording photographically or otherwise the interference of monocromatic light scattered off the surface with a same color reference beam. Of course, if the surface is reentrant then the interference pattern must be recorded for various orientations of the illuminating beam in order to capture the entire surface. In order to physically describe a hologram we must introduce a slowly varying "envelope" electric field whose rms. magnitude corresponds to the intensity of the interference pattern. To this end we write the vector potential on the recording surface as a function of position x on the hologram in the form

$$\mathbf{A}(x) = -(i/k)\,\hat{E}\,(x,t)\,\exp i(k_1 z-\omega_1 t) + \text{h.c.}, \qquad (1)$$

where the factor $\exp i(k_1z-\omega_1t)$ represents the rapidly varying phase of the reference and scattered beams. It is straightforward to quantize these fields by substituting expression (1) into the standard radiation gauge lagrangian for the electromagnetic field. One finds the following commutation relations for the Cartesian component electric field operators on the recording surface:

$$[\,\hat{E}(x,t)\,,\ \hat{E}^{+}(x,t)] = C\delta(x-x'), \qquad (2)$$

where C is a constant having units of photons per second.

In classical holography the electric field receives contributions from various points on the surface. In a similar way in quantum holography the electric field operator $\mathbf{E_s}(x,t)$ associated with photons scattered from various points on a surface can be written in the from:

$$\mathbf{E_s}(x,t) = \frac{1}{\sqrt{V}} \sum_{r_s} i \left(\frac{\hbar\omega}{2\,\varepsilon_0} \right)^{1/2} a_k\, e^{ik(x-r)}\,, \qquad (3)$$

where a_k is the photon creation operator for mode k. The quantum properties of a multi-mode electromagnetic fields have been discussed by many authors. We prefer the treatment of Ou, Hong, and Mandel [8]. They introduce two slowly varying Hermitian quadrature operators $E_1(x,t)$ and $E_2(x,t)$:

$$E_1(x,t) = \hat{E}\,(x,t)\, e^{-i\beta} + \hat{E}^{+}(x,t)\, e^{i\beta}$$
$$E_2(x,t) = -i\,\hat{E}(x,t)\, e^{-i\beta} + i\,\hat{E}^{+}(x,t)\, e^{-\beta}$$

(4)

When the total the total electric field at x is the sum of a coherent field $\mathbf{E_0}\exp i(k_1z-\omega_1t+\varphi)$ and a scattered field of the form (3), then it can be shown [6] that the current-current correlation in a photodetector is a linear function of sin2(β-φ) and cos2(β-φ). The linear coefficients can be expressed in terms of correlation functions for the quadrature operators $E_1(x,t)$ and $E_2(x,t)$. As a corollary, when the photoelectric current J(t) can be expressed as a sum over pulses due to the arrival of photons at times t_j , i.e. $J(t) = \sum_j k(t - t_j)$, then the mean square current can be expressed in the form:

$$< (\Delta J)^2 > = \eta \mid E_1 \mid^2 \int_0^\infty k^2(t')dt' + \eta^2 \mid E_1 \mid^2 q^2 < (\Delta E_1)^2 > \tag{5}$$

where η is the quantum efficiency and q is the total charge resulting from one photon. The first term on the right hand side of (5) is just the usual shot noise, and is the only contribution to the noise when coherent light is being used to make the hologram. The second term on the right hand side of (5) contributes when non-classical quantum states of light are used to make the hologram. The most famous of these non-classical states are the "squeezed light states" for which the contours of the Wigner function for the quadrature operators for a single mode are ellipses [9]. If $< (\Delta E_1)^2 >$ in eq. (5) is negative, then the fluctuations in current density will be sub-Poissonian. Thus production of holograms with non-classical states of light is of interest from the point of view of mitigating the effects of photon counting statistics.

To the author's knowledge the first proposal to construct spatial domain holograms with non-classical quantum states of light was that of Kolobov and Sokolov [10]. Their primary interest was in possibility of producing spatial images with sub-Poissonian levels of noise. What they actually calculated was the squeezed state analog of the classical hologram from a point scatterer. In particular, for a frequency doubled laser beam incident on a point like non-linear converter they studied the interference of parametric down converted photons with the original coherent pump beam. The result of greatest interest is their calculation how fluctuations in photodetector current after homodyning with the coherent pump wave would depend on angle. In particular for the noise in the detector photocurrent after homodyning with the coherent pump wave they obtained the following result

$$< J^2 > = < J > \{1 - \eta[\cos^2 \psi\, e^{2r} + \sin^2 \psi\, e^{-2r} -1]\}, \tag{6}$$

where ψ = Reβ is the angle of rotation of the elliptical contours for the Wigner function, and $\exp(\pm r) = | e^{-i\beta} | \pm | e^{i\beta}|$.

Kolobov and Sokolov pointed out that by restricting the ranges of angles and/or phase mismatching it would be possible to emphasize either amplitude or phase squeezing. In the next section we will show that the use of non-classical states of light which are squeezed with respect to phase rather than amplitude is of particular interest in connection with understanding the relationship between multi-channel quantum mechanics and classical holographic reconstructions of a surface..

3. Semi-classical reconstruction of a quantum surface

In principle, it would be straightforward to generalize the simple geometry considered by Kolobov and Sokolov so as to make a squeezed state holographic representation for a fixed curved surface as shown in Fig. 1. For example, one could imagine covering a smooth curved surface with a strong non-linear medium that produces parametrically down converted photons. If the surface were irradiated with a single coherent pump beam, then the squeezed state photons produced at different points of the surface will interfere with the pump beam and produce an interference pattern on the detector plane that contains, in addition to the usual information about the shape of the surface, information about the squeezing parameter β. Actually it has been experimentally demonstrated [11] that the quantum state of squeezed light incident on a detector plane can be completely determined using homodyning techniques. As in ordinary holography the simplest kind of hologram to consider is one arising from the classical superposition of many elementary interference patterns - an elementary pattern in the quantum case being the spatial pattern of intensity fluctuations given in eq. (6). If the second term in eq. 5 is small, then the surface reconstruction can proceed more or less as in classical holography. A perhaps more interesting problem of surface reconstruction arises when the second term in eq. 5 is important; especially in the case of strong phase squeezing. In this case the possibility arises of using the quantum theory of inverse scattering in its reincarnation as a theory of adaptive optics [2] to reconstruct a quantum surface.

The basic idea is that, instead of a fixed classical surface, we consider a deformed surface Σ whose shape is chosen so as to just compensate for small shifts $\varphi(\sigma,t)$ in the optical path length of a coherent light wave incident on a surface at position σ. We proceed by writing down equations which describe the interplay between small deformations in the shape of a surface and changes in the intensity of light on a set of holograms. The first equation supposes that we have a control system that adjusts the absolute position $X(\sigma,t)$ of the surface with sufficient accuracy so that the intensity of light on the hologram at time t and position x is a linear function of the error $e(\sigma,t) = \Delta X(\sigma,t) + \phi(\sigma,t)$; i.e.

$$I(x,t) = I_0(x) + \int_\Sigma d^2\sigma \; B(x,\sigma)e(\sigma,t), \qquad (7)$$

where I(x,t) is the recorded intensity of light on the hologram at time t and position x, and $I_0(x)$ is the recorded intensity on the hologram when the surface Σ is illuminated with a "standard" beam with no imposed variations in phase with respect to position or time. In writing equation (7) we are neglecting the light propagation time between the surface and the hologram.

The second equation relates the deformation of the surface at point σ, $\Delta X(\sigma,t)$, produced by the feedback control to the intensity of light on the hologram:

$$\Delta X(\sigma,t) = \int d\Omega \int_{-\infty}^{t} d\, t' \; A(\sigma,x,t')\, I(x,t), \qquad (8)$$

where the integral over $d\Omega$ means sampling the light intensity of the phase squeezed light at a sufficiently large number of points on the hologram as is required to determine the parameters τ_1, τ_2,... which define the shape of the surface. When photon counting noise is neglected; i.e. when we regard the hologram as a classical object, then equations (7) and (8) have the solution (in matrix shorthand)

$$e = [1- AB]^{-1} \; \phi. \qquad (9)$$

Equation (9) shows that when the negative feedback is strong the error $e(\sigma,t)$ can be reduced to a small fraction of the perturbation in path length. That is, the position of the surface just tracks locally the change in optical path length. Upon reflection one soon realizes that this just the classical principle of least action! What is perhaps most remarkable about this setup though is that the problem of changing the shape of the surface to compensate for changes in the optical path of the illuminating beam becomes equivalent to solving the multi-channel Schrodinger equation when the effects of photon noise on the hologram are taken into account. This situation is qualitatively different from the classical case because the negative feedback in equation (9) will amplify the photon noise. Therefore unless the photon noise is suppressed, there is now a limit to how strong the negative feedback can be made.

It is not hard to show that in the presence of photon noise the two point correlation function for the path length errors $e(\sigma,t)$ averaged over a time long compared to the characteristic time for photon number fluctuations has the form (again in operator notation):

$$< e_1\, e_2> = [1-A_1\, B_1]^{-1}\, [1-\, A_2\, B_2]^{-1}\, \{U_{12} + A_1\, A_2\, \delta_{12}\, I_0\}, \qquad (10)$$

where U_{12} is the average $<\varphi_1\varphi_2>$ over the same time and $\delta_{12} = \delta(\sigma_1 - \sigma_2)\, \delta\, (t_1 - t_2)$. In contrast with the classical case, an optimal choice for the feedback matrix A is somewhat arbitrary. However, following Dyson [2], we may take as the criterion for optimizing the feedback system that a quadratic function of the feedback errors should be minimized. Then it can be shown that the optimal feedback matrix $A(\sigma,x,t')$ can be expressed in the form:

$$A = KB^T\, I_0^{-1}$$

(11)

where $K(\sigma_1\, ,\, \sigma_2\, ,\, t_1 - t_2)$ is a causal matrix satisfying the non-linear operator equation

$$K + K^T\; + K(B^T\, I_0^{-1}B)K^T\; + U = 0 \qquad (12)$$

This equation is a matrix generalization of the Gelfand - Levitan equation, and is essentially equivalent to the multi-channel inverse scattering theory of Newton and Jost [12]. In our quantum optical realization of their theory the discrete channel labels represent a set of spatially localized illuminations of the surface.

The similarity of the feedback control equations (11) and (12) to the inverse scattering equations for multi-channel quantum mechanics can be taken to mean that our scheme for using the photon counting correlations of a phase squeezed state hologram to control the shape of a surface provides a kind of quasi-classical reconstruction for a quantum mechanical surface. In particular, the holographic equation (8) would allow one to correlate the position of the surface with an asymptotic phase. Of course, such a feedback system would not actually generate a quantum mechanical object in the sense that that at any given time the deformed surface has a well defined shape. However, by correlating the asymptotic phases derived from a sequence of shapes for the surface, one could over a period of time reconstruct what is in effect a semi-classical quantum mechanical "wavefunction" for a deformed surface.

4. Quantum Riemann surfaces

As a generalization of the familiar elementary problem of quantizing a flat two dimensional phase space parameterized by p and q, we now turn to the problem of quantizing the parameterizations of a smooth curved two dimensional surface Σ. When Σ is a complex algebraic curve, i.e. a Riemann surface, the surface can be represented by a collection of flat sheets connected together along branch cuts [13]. Thus in this case the problem of quantizing parameterizations of the surface would appear to be very similar to the original Wigner-Moyal problem of quantizing flat p,q phase space, except that now the phase space quantizations on each sheet must be matched along the branch cuts. One might guess that such a system could be quantized by introducing a set of p,q variables with an index j which represented which sheet one was on. The photon annihilation and creation operators $\mathbf{E}$ and $\mathbf{E}^+$ introduced in section 2 are now replaced with sets of annihilation and creation operators $\{\mathbf{a}_j = \mathbf{p}_j + i\mathbf{q}_j\}$ and $\{\mathbf{a}^+_j = \mathbf{p}_j - i\mathbf{q}_j\}$. The original Weyl-Heisenberg group will be replaced by a Lie group generated by operators of the form $\alpha^* \mathbf{a} \equiv \sum_{j=1}^{N} \alpha_j^* \mathbf{a}_j$ and $\alpha\, \mathbf{a}^+ \equiv \sum_{j=1}^{N} \alpha_j \mathbf{a}^+_j$. States playing much the same role as the Glauber coherent states introduced in section 2 will be generated by the operators

$$\mathbf{D}(\alpha) = \exp(\alpha\, \mathbf{a}^+ - \alpha^* \mathbf{a}). \tag{13}$$

which are analogous to the displacement operators $\mathbf{D}(\xi,\eta)$.These operators obey the multiplication rule

$$\mathbf{D}(\alpha)\mathbf{D}(\beta) = e^{i\mathrm{Im}(\alpha\beta^*)}\mathbf{D}(\alpha+\beta). \tag{14}$$

Replacing the Weyl- Heisenberg algebra with an arbitrary Lie algebra leads to the generalized coherent states of Gilmore and Perelomov [14]. For these generalized coherent states the space parameterized by α and α^* is no longer the complex plane but a symmetric space G/H. These symmetric spaces have a natural symplectic structure, and moreover, provide a natural phase space structure for classical non-linear dynamical systems whose dynamics is described by Lax-pair type equations [15]. For example, in the SU(N) case in terms of the variables

$$z = \alpha \frac{\sinh \alpha\alpha^*}{\sqrt{\alpha\alpha^*}}$$

the metric of the space parametrized by the α variables is proportional to $dzdz^*$, and the Poisson bracket takes the simple form

$$\{f,g\} = -i\sum_j \left[\frac{\partial f}{\partial z_j^*}\frac{\partial g}{\partial z_j} - \frac{\partial g}{\partial z_j^*}\frac{\partial f}{\partial z_j} \right]. \tag{15}$$

Evidently the variables z and z^* play essentially the same role as the p and q variables in ordinary mechanics. Therefore we expect that the Wigner-Moyal quantization procedure [16] can be used to define operators on the 2N-dimensional phase space parameterized by α and α^* and introduce a Wigner distribution function on this space.

It is interesting to note that when G/H is a Jacobian variety then the coherent states generated by the $\mathbf{D}(\alpha)$ can be identified with the theta function that plays such an important role in the theory of Riemann surfaces and their moduli [7]. Indeed the condition that the generalized coherent states be single valued on the torus means that the phase factor $\mathrm{Im}(\alpha\beta^*)$ in equation (14) must be equal to 2π times an integer when α and β correspond to periods of the torus. Remarkably this is just the condition that a complex N-torus also has an interpretation as the Jacobian variety of a Riemann surface [cf. 7]. Thus it is possible to regard the

Jacobian variety of a Riemann surface as a kind of classical phase space and to use theta functions to define a quantization of this phase space a la Wigner- Moyal. Since according to the classical Torelli theorem a Riemann surface can be reconstructed from its Jacobian variety and associated theta functions [7], one might assume that applying the Wigner-Moyal formalism to quantize the Jacobian variety of a fixed Riemann surface effectively solves the problem of defining a "quantized Riemann surface". Alternatively, since in the case of a flat surface Wigner-Moyal quantization is equivalent to the usual Heisenberg-Born-Jordan quantization, what is involved here is presumably an assumption that introducing a deformed Riemann surface as a quantum object is equivalent to replacing a set of classical holographic representations of a Riemann surface with non-classical holograms.

The scattered electric field $\mathbf{E_s}$(x,t) receives contributions from various points on the surface. In the dipole approximation the contribution from each little patch of surface is determined by the cross product between the vector pointing from the patch of surface to the point x and the direction of the oscillating polarization induced in the surface patch by the illuminating beam. Since these two vectors are curl-free on the surface the sum of contributions over the surface has the character of an inner product of two harmonic differentials:

$$(\omega_1 , \omega_2) = \int_\Sigma \omega_1 \wedge \omega_2{}^* \qquad (16)$$

Now an inner product for harmonic 1-forms of the form (16) plays an important role in the Torelli theorem reconstruction for a Riemann surface [cf. ref.7], so we here have evidence of a close connection between reconstructions of a Riemann surface from its period matrix and holographic representations of the Riemann surface.

In one obviously important respect, though, the quantization of the holographic electric field $\mathbf{E}$(x,t) discussed in section 2 appears to differ from the symmetric space quantization that involves g-copies the Weyl-Heisenberg group; namely, the holographic representation seems to involve an infinite number of annihilation and creation operators corresponding to an infinite number of positions on the hologram, whereas the quantization based on the Jacobian variety involves only a finite number of such operators. This discrepancy can be traced to the fact that the quantization problem most closely related to ordinary p,q phase space quantization assumes that the shape of the surface is determined by just a few parameters. Under such circumstances the electric field vectors at different points on the detector plane are not independent for a given orientation of the illuminating beam. Indeed upon reflection it is clear that in the case of a Riemann surface the only way to obtain algebraically independent annihilation and creation operators is to illuminate the different topological handles of the Riemann surface in distinctive ways. Indeed, it is physically clear that, at least in the short wavelength limit, two different patterns of illumination are needed in order to represent each handle.

An aesthetic choice for the independent illuminations needed for each handle would be to choose the different illuminating beams in such a way that the polarizations induced on the surface by the different beams can be identified with some canonical basis for the first cohomology group of the surface. Physically this corresponds to choosing a set of illuminations for the surface such that the polarizations of the oscillating electric currents induced on the surface by the illuminating beams can be identified with a set of independent 1-forms ω_j. There are 2g independent real 1-forms in such a basis, whose support can be taken to coincide with the support of harmonic functions f_j such that $\omega_j = df_j$ [13]. On the other hand, since we are interested in smoothly deforming the surface in such a way that the deformed surface approximates a Riemann surface, it might be more useful to combine illuminations of the surface corresponding to real periods on the surface into holomorphic complex 1-forms. In fact, there are g independent holomorphic 1-forms on a Riemann surface of genus g [13]. Therefore utilizing “holomorphic” illuminations corresponding to using holomorphic 1-forms in eq. (16), one could record a set of g independent holograms from which the Riemann surface could in principle be completely reconstructed.

Following the quantization program of section 2, one can associate independent photon annihilation and creation operators, $\hat{E}_{\mathrm{i}}$ and $\hat{E}_{\mathrm{I}}^{+}$, with each independent hologram, thus corroborating our previous guess for the structure of a phase space corresponding to a Riemann surface. Taking into account the quantum fluctuations associated with these 2g degrees of freedom would lead to what might be legitimately interpreted as a quantized Riemann surface, where the classical coordinates of the Riemann surface have *in some sense* been replaced with non-commuting operators. This is the point of view that has been adopted for a quantum theory of membranes [4]. On the other hand, when restricted to a space where the photon number is zero or one, the Fock space annihilation and creation operators can be interpreted as spin operators. Thus in the low intensity limit one may describe quantized holographic representations of a Riemann surface in terms of a Hilbert space for g qubits.

As noted in section 2 one can use 2-photon counting correlations as measured on a planar array of photon detectors to back out the quantum amplitudes describing the non-classical states of light being generated on a surface by one or more phase coherent illuminating beams as shown in Fig. 1. This would allow one to assign a Fock space wavefunction to each independent illumination of the Riemann surface. In the low intensity limit these Fock space wavefunctions describe qubits, so that roughly speaking there would be a qubit associated with each handle on the surface. If a single laser beam is used to illuminate the whole surface then the quantum surface can be described by a direct product of the qubits associated with the topological handles. A perhaps more interesting situation would arise, though, if one could construct a quantum superposition of qubit states corresponding to different views of the Riemann surface. This would provide an elegant solution to the familiar practical problem of meshing together holographic views of a reentrant surface obtained from different perspectives, and at the same time shows that entangled quantum states can be used to characterize the global topology of certain moduli spaces.

5. Entangled state representations for quantum surfaces

In the low intensity limit, the most general wavefunction for a quantum Riemann surface that could be constructed by superimposing quantum holograms of the type discussed in section 2 will have the form [5]

$$\Psi(x) = \sum_{\{n_j\}} a(x, n_1, n_2, \ldots n_N) \prod_{i=1}^{g} |n_j\rangle \qquad (17)$$

where x is position on the detector plane and the n_j =0,1 are the squeezed state photon numbers associated with canonical “holomorphic” illuminations of the surface. For fixed x, each term in (17) characterizes the shape of the template surface as viewed from a certain perspective and with a certain orientation of the illuminating beam. An interesting question is how to use all the terms in (17) to the reconstruct the global geometry of what is now a quantum Riemann surface. As a step in this direction one might hope to combine measurements of spatial and temporal photon counting correlations with a feedback scheme of the same nature as the adaptive optics scheme discussed in section (3) to relate changes in the wavefunction (17) to changes in the geometry of the quantum surface.

Unfortunately, there seem to be fundamental barriers to using a semi-classical scheme like that described in section 3 to reconstruct the quantum surface associated with an entangled wavefunction. Even with the use of sophisticated measurements like those described in ref. 11, reconstruction of a quantum surface in a manner similar to that described in section 3 would require not only the classical recording of time domain photon counting correlations, but also numerical computations to determine the changes in the pump beam parameters. Thus despite the use of non-classical states of light, a new quantum state would have to generated using classical information. Moreover, using the holographic scheme illustrated in Fig. 1 it could only be generated 1 qubit at a time. As is well known [17] it is not possible (or at least not known how) to generate arbitrary states of a quantum mechanical system using classically controlled single qubit operations. In order to go beyond the limitations of classical data bases and single qubit operations, one must evidently employ multi-qubit purely quantum manipulations.

It might be noted that the non-classical holograms discussed in section 2 make use of only one of the down converted pair of photons created from each limited area of the surface (cf. Fig. 1). However the second photon is entangled with the first photon and is a quantum resource that one can use. Indeed, by making use of the second entangled photon associated with each qubit one ought to be able to make an exact replica of the original hologram using quantum teleportation [18]. The basic point is that the propagation of down converted photons from various points on the surface to a particular point on the detector plane creates a state that is a product of qubits, each of which is one of an EPR pair. The measurement techniques discussed in ref. 11 would allow one to measure the hologram qubits. Remarkably measuring one qubit in an EPR pair is just the setting required for teleportation of single qubits. Thus one could use single qubit teleportation and transformation techniques to reconstruct any quantum surface described by a product of qubits.

Actually it has recently been pointed out [19] that one ought to be able to use multi-photon teleportation techniques to create an arbitrary entangled state of multiple qubits, provided 3-photon correlated states are available.. Such states are not yet readily available in the laboratory, but there is some evidence [20] they can be produced. Thus, production of a quantum superposition of squeezed state holograms corresponding to different views of the template surface and using multi-photon entangled states may soon be not just a conceptual but also practical possibility.. The wavefunction for a multi-photon quantum hologram could in principle be measured using the same kind of multi-port homodyning techniques [21] that have already seen extensive use in quantum optics experiments. Therefore, we may readily conceive of a thought experiment in which the unused photons in the Kolobov and Sokolov scheme are used to generate a virtual multi-photon state representing the global geometry of a quantum surface.

Describing the geometry and topology of the virtual quantum surface associated with the wavefunction (17) is an intriguing mathematical problem. Because of quantum fluctuations in the classical amplitude and phase, each term in (17) will correspond to a family of deformations of a Riemann surfaces. The superposition of the various terms in (17) apparently describes a ensemble of such foliations, all of the same topological type. It is worth noting in this connection that the feedback equations (7) and (8) imply that an entangled quantum hologram defines a flat $U(2^g)$ bundle over the Riemann surface and this bundle is equivalent to an ensemble of families of deformations of the Riemann surface. As it happens both the algebra for combining representations of flat U(N) connections over a Riemann surface and topological invariants for families of deformations of Riemann surfaces played an important role in the development of quantum cohomology [3]. Thus it is quite possible that our elementary derivation of a connection between entangled states of qubits and the global geometry and topology of families of deformations of Riemann surfaces is connected with and perhaps even underlies these sophisticated mathematical developments.

Another interesting question concerns the relationship of our entangled qubit representation for a quantum surfaces with fundamental physics. Our qubit representation for a quantum Riemann surface might be considered to be a simple model for M-theory [4]. On the other hand, ensembles of families of Riemann surfaces play an important role in both curved twistor spaces and Kummer surfaces, which may play an important role in the Planck scale structure of space-time [22]. This suggests that one should consider the possibility that the fundamental degrees of freedom in physics can be described in terms of multi-qubit entangled states. For example, a Kummer surface presumably corresponds to a special 24 qubit state, which could then be paired via entanglement with another 24 qubit state representing the dual Kummer surface. Thus the fundamental objects in a theory of everything may simply multi-qubit generalizations of the familiar 2-qubit EPR pairs.

6. Summary

We have considered the ways in which both unentangled and entangled non-classical states of light might be used to holographically represent the quantum fluctuations of a deformable surface. These constructions have concrete realizations based on 1) the use of coherent pump waves to produce non-classical EPR photon states on the surface, 2) the use of homodyning techniques for measuring quantum wavefunctions, and 3) the use of quantum teleportation techniques to entangle quantum holographic states. In the case of a deformable Riemann surface with genus >1, these constructions apparently provide a means to connect topologically non-trivial 4-manifolds with entangled qubit wavefunctions. This connection sheds new light on the relationship between mathematics and physics, and may play an important role in fundamental physics.

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Figure 1

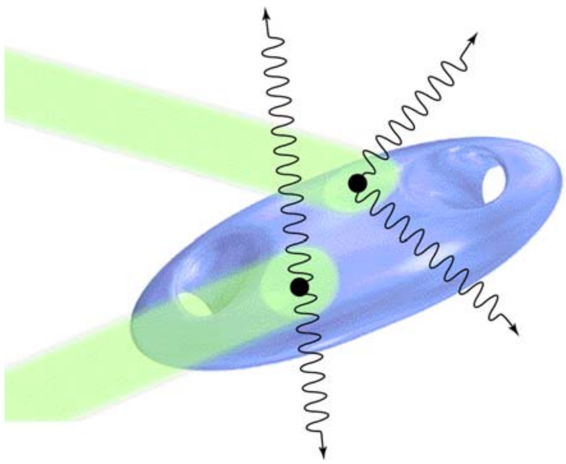

Figure caption

Production of squeezed state photons by parametric down conversion on a smoothly curved surface coated with a non-linear material.